EXACT SOLUTIONS IN MULTIDIMENSIONAL COSMOLOGY WITH SHEAR AND BULK VISCOSITY

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Abstract

Multidimensional cosmological model describing the evolution of a fluid with shear and bulk viscosity in n Ricci-flat spaces is investigated. The barotropic equation of state for the density and the pressure in each space is assumed. The second equation of state is chosen in the form when the bulk and the shear viscosity coefficients are inversely proportional to the volume of the Universe. The integrability of Einstein equations reads as a colinearity constraint between vectors which are related to constant parameters in the first and second equations of state. We give exact solutions in a Kasner-like form. The processes of dynamical compactification and the entropy production are discussed. The non-singular D-dimensional isotropic viscous solution is singled out.

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1 Introduction

Last two decades have witnessed an increase of interest to the multidimensional cosmology (see, for instance, [5-7,9,11-18,20-25,27-29,31,32,39,41,42,45-48] and references therein). According to this theory one assumes that the Universe had a higher dimension at a very early stage of its evolution, and that quantum processes have been responsible for the (topological) partition of the space, which provides us at present with the usual 3-dimensional (*external*) space, in addition to *internal* space(s). The manifold which accounts for such a multidimensional spacetime has the following topology

$$M = R \times M_1 \times \ldots \times M_n, \tag{1.1}$$

where R stands for the cosmic time axis, and the product with one part of manifolds $M_1, \ldots, M_n$ gives the external space, when the remaining part stands for internal spaces. The classical stage of the evolution is governed by the multidimensional version of Einstein's equations. According to the classical description, the Riemann curvatures of spaces $M_1, \ldots, M_n$ are assumed to be constant (Einstein spaces). Such a model is the simplest multidimensional generalization of the space-time upon which the Friedman-Robertson-Walker (FRW) world model is based on. One easily understands that the use of extra dimensions (for the physical space-time) can be a sensible scenario only at primordial epochs, since the standard FRW world model is known to be in a sufficiently good agreement with the observational constraints down to a quite primordial epoch such as the nucleosynthesis era. Hence, it is clear that a reduction process (called *dynamical compactification* of additional dimensions) is required before such an epoch, to make the internal spaces contracting themself down to unobservable sizes.

Herein, we assume that the cosmic fluid (the source of the gravitational field at early stages) is viscous, which might simulate high energy physics processes (such as the particles creation). The effects related to viscosity in 4-dimensional Universe were studied through different viewpoints (see e.g., [2-4,8,10,26,30,33-38,44]). Before developing the multidimensional model let us briefly discuss (extensive review of the subject was given by Gron [19]) the main trends in 4-dimensional cosmology with viscous fluid as a source.

First, Misner [33] considered neutrino viscosity as a mechanism of reducing the anisotropy in the Early Universe. Stewart [40] and Collins and Stewart [10] proved that it is possible only if initial anisotropies are small enough. Another series of papers which concerns the production of entropy in the viscous Universe was started by Weinberg [44]. Both isotropization and production of entropy during lepton era in models of Bianchi types I,V were considered by Klimek [26]. Caderni and Fabbri [8] calculated coefficients of shear and bulk viscosity in plasma and lepton eras within the model of Bianchi type I. The next approach is connected with obtaining singularity free viscous solutions. The first nonsingular solution was obtained by Murphy [35] within the flat FRW model with fluid possessing a bulk viscosity. However, Belinsky and Khalatnikov [2, 3] showed that this solution corresponds to the very peculiar choice of parameters and is unstable with respect to the anisotropy perturbations. Other nonsingular solutions with bulk viscosity were obtained by Novello and Araujo [36], Romero [38], Oliveira and Salim [37].

The crucial feature of each viscous cosmological model is assuming of the so called "second equation of state", which provides us with the viscosity coefficients dependence on time. Further we denote by ζ and η the bulk and shear viscosity coefficients, correspondingly. Murphy [35] integrated the 4-dimensional flat FRW model with bulk viscosity by assuming $\zeta \sim \rho$ (as second equation of state), where ρ is the density of the viscous fluid. Belinsky and Khalatnikov [2, 3] studied the behavior of this model as well as homogeneous anisotropic models of Bianchi types I and IX by means of qualitative methods with more general second equations of state $\zeta, \eta \sim \rho^{\nu}$, where ν is constant. Lukacs [30] integrated the homogeneous and isotropic 4-dimensional model with a viscous pressureless fluid and a second equation of state given by $\zeta \sim$ [scale factor]$^{-1}$. A curvature-dependent bulk viscosity was studied in multidimensional cosmology by Wolf [47]. Recently, Motta and Tomimura [34] studied a 4-dimensional inhomogeneous cosmology with a bulk viscosity coefficient which depends on the metric. In our previous papers [15, 17] exact solutions in multidimensional models with bulk viscosity were obtained and their properties were studied for the following type of the second equation of state: $\zeta \sim$ [volume of the Universe]$^{-1}$.

The aim of the present investigation is to integrate the Einstein equations for a multidimensional cosmological model formed by a chain of Ricci-flat spaces and a cosmic fluid possessing both shear and bulk viscosity. The second equations of state are chosen in the following form of metrical dependence of the bulk and shear viscosity coefficients: $\zeta, \eta \sim$ [volume of the Universe]$^{-1}$.

The paper is organized as follows. In Sec. 2 we describe the model and get basic equations. To integrate them, we develop some vector formalism proposed in our previous papers [14, 16]. Thermodynamical concepts in multidimensional cosmologies are defined in Sec. 3, where a formula which provides us with the variation rate of the entropy is derived. The equations of motion for the special set of parameters in the first and the second equations of state are integrated in Sec. 4. Exact solutions are given in a Kasner-like form, their physical properties are investigated in Sec. 5, where the process of dynamical compactification and the entropy production are also defined.

2 The model

Let us have the following metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\gamma(t)}\mathrm{d}t^2 + \sum_{i=1}^{n} \exp[2x^i(t)]\mathrm{d}s_i^2, \tag{2.1}$$

on the manifold defined in Eq. (1.1), where $\mathrm{d}s_i^2$ is the metric of the Einstein space M_i, $\gamma(t)$ and $x^i(t)$ are scalar functions of the cosmic time t. The dimension of this manifold is given by $D = 1 + \sum_{i=1}^{n} N_i$, where $N_i = \dim M_i$. Herein, for reason of simplicity, only Ricci-flat spaces $M_1, \ldots, M_n$ are assumed (i.e., the components of the Ricci tensor for the metrics $\mathrm{d}s_i^2$ are zero). One easily obtains the non-zero components of the Ricci-tensor for the metric defined in Eq. (2.1) (see [23]):

$$R_0^0 = \mathrm{e}^{-2\gamma(t)} \left(\sum_{i=1}^{n} N_i (\dot{x}^i)^2 + \ddot{\gamma}_0 - \dot{\gamma}\dot{\gamma}_0 \right), \tag{2.2}$$

$$R_{n_i}^{m_i} = \mathrm{e}^{-2\gamma(t)} \left(\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i \right) \delta_{n_i}^{m_i}, \tag{2.3}$$

for $i = 1, \ldots, n$, where the indices m_i and n_i run from $D - \sum_{j=i}^{n} N_j$ to $D - \sum_{j=i}^{n} N_j + N_i$ and $\gamma_0 = \sum_{i=1}^{n} N_i x^i$.

A viscous fluid is characterized by a density ρ, a pressure p, a bulk viscosity coefficient ζ, a shear viscosity coefficient η, so that the (standard form of the) energy-momentum tensor reads

$$T_\nu^\mu = \rho u^\mu u_\nu + (p - \zeta\theta) P_\nu^\mu - 2\eta\sigma_\nu^\mu, \tag{2.4}$$

where u^μ is the D-dimensional velocity of the fluid, $\theta = u^\mu_{;\mu}$ denotes the scalar expansion, $P_\nu^\mu = \delta_\nu^\mu + u^\mu u_\nu$ is the projector on the $(D-1)$-dimensional space orthogonal to u^μ, and $\sigma_\nu^\mu = \frac{1}{2}\left(u_{\alpha;\beta} + u_{\beta;\alpha}\right) P^{\alpha\mu} P_\nu^\beta - (D-1)^{-1}\theta P_\nu^\mu$ is the traceless shear tensor (defined as usual).

By choosing the D-dimensional velocity so that $u^\mu = \delta_0^\mu \mathrm{e}^{-\gamma(t)}$ (the comoving observer condition), we obtain

$$\theta = \dot{\gamma}_0 \mathrm{e}^{-\gamma(t)}, \tag{2.5}$$

$$(u^\mu u_\nu) = \mathrm{diag}(-1, 0, \ldots, 0), \tag{2.6}$$

$$(P_\nu^\mu) = \mathrm{diag}(0, 1, \ldots, 1), \tag{2.7}$$

$$(\sigma_\nu^\mu) = \mathrm{e}^{-\gamma}\mathrm{diag}\left(0, \left(\dot{x}^1 - \frac{\dot{\gamma}_0}{D-1}\right)\delta_{l_1}^{k_1}, \ldots, \left(\dot{x}^n - \frac{\dot{\gamma}_0}{D-1}\right)\delta_{l_n}^{k_n}\right), \tag{2.8}$$

where $k_i,\ l_i = 1,\dots,N_i$ for $i = 1,\dots,n$. The function $\gamma(t)$ determines a time gauge (a harmonic time gauge for $\gamma(t) = \gamma_0$ and a synchronous time gauge for $\gamma(t) = 0$), see Eq. (2.1); note that the harmonic time t and the synchronous time t_s are related by $\mathrm{d}t_s = \exp[\gamma_0]\mathrm{d}t$. By assuming anisotropy properties for the pressure and the bulk viscosity, with respect to the whole space $M_1 \times \ldots \times M_n$, one has

$$(T^\mu_\nu) = \mathrm{diag}(-\rho, p^*_1\delta^{k_1}_{l_1},\dots,p^*_n\delta^{k_n}_{l_n}), \tag{2.9}$$

where

$$p^*_i = p_i - \mathrm{e}^{-\gamma}\left[\zeta_i\dot\gamma_0 + 2\eta\left(\dot x^i - \frac{\dot\gamma_0}{D-1}\right)\right], \tag{2.10}$$

and p_i, resp. ζ_i, is the pressure, resp. the bulk viscosity coefficient, in the space described by the manifold M_i. Furthermore, we assume that the barotropic equations of state hold

$$p_i = (1-h_i)\rho(t), \tag{2.11}$$

where the h_i are constants. One easily shows that the form of the equation of motion ($\bigtriangledown_M T^M_0 = 0$) for a viscous fluid described by a tensor given by Eq. (2.9), is given by

$$\dot\rho + \sum_{i=1}^n N_i\dot x^i(\rho + p^*_i) = 0. \tag{2.12}$$

The Einstein equations $R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \kappa^2 T^\mu_\nu$, where κ^2 is the gravitational constant, can be written as $R^\mu_\nu = \kappa^2(T^\mu_\nu - \frac{T}{D-2}\delta^\mu_\nu)$. Further, by using the equations $R^0_0 - \frac{1}{2}\delta^0_0 R = \kappa^2 T^0_0$ and $R^{m_i}_{n_i} = \kappa^2(T^{m_i}_{n_i} - \frac{T}{D-2}\delta^{m_i}_{k_i})$, Eqs.(2.2,2.3,2.9) give the following equations of motion

$$\sum_{i=1}^n N_i(\dot x^i)^2 - \dot\gamma_0^2 = -2\kappa^2\mathrm{e}^{2\gamma}\rho, \tag{2.13}$$

$$\begin{aligned}\ddot x^i + (\dot\gamma_0 - \dot\gamma)\dot x^i &= \kappa^2\mathrm{e}^\gamma\left[\mathrm{e}^\gamma\rho\left(-h_i + \frac{\sum_{k=1}^n N_k h_k}{D-2}\right)\right.\\ &+ \left.\dot\gamma_0\left(-\zeta_i + \frac{\sum_{k=1}^n N_k\zeta_k}{D-2}\right)\right] - 2\kappa^2\mathrm{e}^\gamma\eta\left(\dot x^i - \frac{\dot\gamma_0}{D-1}\right).\end{aligned} \tag{2.14}$$

We use an integration procedure which is based on the n-dimensional real vector space R^n. Let $e_1,\dots,e_n$ be the canonical basis in R^n (i.e. $e_1 = (1,0,\dots,0)$ etc...), and $\langle,\rangle$ denote a symmetrical bilinear form defined on R^n by

$$\langle e_i, e_j\rangle = \delta_{ij}N_j - N_iN_j \equiv G_{ij}. \tag{2.15}$$

It has been used as a mini-super-space metric for cosmological models (see [20-25]). Such a form is non-generate and has the pseudo-Euclidean signature $(-,+,\dots,+)$. With this in mind, a vector $y \in R^n$ is time-like, resp. space-like or isotropic, if $\langle y, y\rangle$ takes negative, resp. positive or null values; and two vectors y and z are orthogonal if $\langle y, z\rangle = 0$. Hereafter, we use the following vectors

$$x = x^1e_1 + \ldots + x^ne_n, \tag{2.16}$$

$$u = u^1e_1 + \ldots + u^ne_n, \quad u^i = h_i - \frac{\sum_{k=1}^n N_kh_k}{D-2}, \quad u_i = N_ih_i, \tag{2.17}$$

$$\xi = \xi^1e_1 + \ldots + \xi^ne_n, \quad \xi^i = \zeta_i - \frac{\sum_{k=1}^n N_k\zeta_k}{D-2}, \quad \xi_i = N_i\zeta_i, \tag{2.18}$$

where covariant coordinates of the vectors are introduced by the usual way. Moreover, let us denote u_d the particular vector given by Eq. (2.17) with $h_{i=1,\ldots,n} = 1$ (it is related to dust in the whole space, see Eq. (2.11)). One has

$$(u_d)_i = N_i, \quad u_d^i = \frac{-1}{D-2}, \quad \langle u_d, u_d \rangle = -\frac{D-1}{D-2}, \quad \langle u_d, x \rangle = \gamma_0. \tag{2.19}$$

Thus, using Eqs. (2.16-2.19) we rewrite the Einstein equations (2.13),(2.14) in the form

$$\langle \dot{x}, \dot{x} \rangle = -2\kappa^2 \mathrm{e}^{2\gamma} \rho, \tag{2.20}$$

$$\ddot{x} + \left(\langle u_d, \dot{x} \rangle - \dot{\gamma} + 2\eta\kappa^2 \mathrm{e}^{\gamma} \right) \dot{x} = \frac{\langle \dot{x}, \dot{x} \rangle}{2} u - \kappa^2 \mathrm{e}^{\gamma} \langle u_d, \dot{x} \rangle \left(\xi - \frac{2\eta}{\langle u_d, u_d \rangle} u_d \right), \tag{2.21}$$

where the formal dependence on ρ in Eq. (2.21) has been canceled, according to Eq. (2.20). Moreover, Eq. (2.12) can be written as

$$\dot{\rho} + \langle 2u_d - u, \dot{x} \rangle \rho - \mathrm{e}^{-\gamma} \left(2\eta \langle \dot{x}, \dot{x} \rangle + \langle u_d, \dot{x} \rangle \langle \xi - \frac{2\eta u_d}{\langle u_d, u_d \rangle}, \dot{x} \rangle \right) = 0. \tag{2.22}$$

To integrate Eq.(2.21) one needs second equations of state, involving the bulk viscosity coefficients $\zeta_1, \ldots, \zeta_n$ and the shear viscosity coefficient η. Let us assume that these coefficients are proportional to $\exp[-\gamma_0]$ (or inversely proportional to the volume of the Universe), i.e.

$$\eta, \zeta_i \sim [\text{scale factor of } M_1]^{-\dim(\mathrm{M}_1)} \cdot \ldots \cdot [\text{scale factor of } M_n]^{-\dim(\mathrm{M}_n)}, \tag{2.23}$$

which means (from a physical viewpoint) that the viscosity decreases when the space $M_1 \times \ldots \times M_n$ expands. The integrability of the basic equation (provided the second equations of state) is ensured when the vectors u, ξ, u_d are either colinear or orthogonal (with respect to the mini-super-space metric) in some combination [15,18]. Herein, we suppose that these vectors are colinear, which means that the viscous fluid has identical properties in the internal space(s) and the external space. Hence, all these assumptions, for the pressures and the viscosity coefficients, allow us to write

$$p_i = (1-h)\rho \quad \text{or} \quad u = hu_d, \tag{2.24}$$

$$\zeta_i = \frac{\zeta_0}{\kappa^2} \mathrm{e}^{-\gamma_0} \equiv \zeta \quad \text{or} \quad \xi = \zeta u_d = \frac{\zeta_0}{\kappa^2} \mathrm{e}^{-\gamma_0} u_d, \quad (i = 1, \ldots, n), \tag{2.25}$$

$$\eta = \frac{\eta_0}{2\kappa^2} \mathrm{e}^{-\gamma_0}, \tag{2.26}$$

where ζ_0, η_0 and h are constants.

3 Multidimensional Thermodynamics

According to [43,48], let us summarize thermodynamics principles in such a multidimensional cosmology. The first law of thermodynamics reads

$$T\mathrm{d}S = \mathrm{d}(\rho V) + V \sum_{i=1}^{n} p_i \frac{\mathrm{d}V_i}{V_i}, \tag{3.1}$$

where V_i stands for a fluid volume in the space M_i, when $V = V_1 \cdot \ldots \cdot V_n$ is a fluid volume in the whole space, and S is an entropy in the volume V. By assuming that the baryon particle number N_{B} in the volume V is conserved, Eq. (3.1) transforms to

$$nT\dot{s} = \dot{\rho} + \rho \sum_{i=1}^{n} N_i \dot{x}^i + \sum_{i=1}^{n} p_i N_i \dot{x}^i, \tag{3.2}$$

where $s = S/N_{\mathrm{B}}$, resp. $n = N_{\mathrm{B}}/V$, stands for the entropy per baryon, resp. the baryon number density. Let us remind that $\exp[x^i]$ is the scale factor of the space M_i (of dimension N_i). For a perfect fluid ($\zeta_i = 0$, $\eta = 0$), the comparison between Eqs. (2.12,3.2) gives the entropy conservation (i.e., s is constant). Similarly, we obtain also the temperature (see [48]). From Eq. (3.2) we have

$$\left(\frac{\partial \rho}{\partial x^i}\right)_{s,x^j} = -\rho N_i - p_i N_i = (h_i - 2) N_i \rho, \quad (j \neq i), \tag{3.3}$$

and then

$$\rho = K(s) \exp\left[\sum_{i=1}^{n} (h_i - 2) N_i x^i\right], \tag{3.4}$$

where $K(s)$ is an unknown function (which reads in term of the entropy per baryon s). Using Eqs. (3.2),(3.4) we get

$$\left(\frac{\partial \rho}{\partial s}\right)_{x^i} = nT = K'(s) \exp\left[\sum_{i=1}^{n} (2 - h_i) N_i x^i\right]. \tag{3.5}$$

For a perfect fluid, we have $K'(s) = 1/B$ where B is a constant, then

$$nT = \frac{1}{B} \exp\left[\sum_{i=1}^{n} (h_i - 2) N_i x^i\right] = \frac{1}{B} \exp[\langle u - 2u_d, x\rangle]. \tag{3.6}$$

For a fluid with a bulk and shear viscosity, the comparison between Eqs. (2.12,3.2) provides us with

$$nT\dot{s} = \mathrm{e}^{-\gamma} \left(2\eta \langle \dot{x}, \dot{x}\rangle + \langle u_d, \dot{x}\rangle \langle \xi - 2\frac{2\eta}{\langle u_d, u_d\rangle} u_d, \dot{x}\rangle\right). \tag{3.7}$$

Such a formula gives the variation rate of entropy per baryon in multidimensional cosmology on the manifold $M = R \times M_1 \times \ldots \times M_n$ with viscosity. The entropy production can be calculated if the temperature of the fluid is known. Herein, we suppose that the temperature is given by the perfect fluid formula Eq. (3.6), which is valid with sufficient accuracy when effects of viscosity are small. Hence, Eqs. (3.6),(3.7) give

$$\dot{s} = B \mathrm{e}^{\langle 2u_d - u, x\rangle - \gamma} \left(2\eta \langle \dot{x}, \dot{x}\rangle + \langle u_d, \dot{x}\rangle \langle \xi - 2\frac{2\eta u_d}{\langle u_d, u_d\rangle}, \dot{x}\rangle\right). \tag{3.8}$$

4 Exact solutions

According to assumptions given in Eqs. (2.24-2.26), the basic vector equation, see Eq. (2.21), reads

$$\ddot{x} + \left(\langle u_d, \dot{x}\rangle - \dot{\gamma} + 2\eta\kappa^2 \mathrm{e}^{\gamma}\right)\dot{x} = \left[\frac{h}{2}\langle \dot{x}, \dot{x}\rangle - \kappa^2 \mathrm{e}^{\gamma}\langle u_d, \dot{x}\rangle \left(\zeta - \frac{2\eta}{\langle u_d, u_d\rangle}\right)\right] u_d. \tag{4.1}$$

In order to integrate such a (vector) equation, we use the orthogonal basis

$$\frac{u_d}{\langle u_d, u_d\rangle}, f_2, \ldots, f_n \in R^n, \tag{4.2}$$

where the orthogonality property reads

$$\langle u_d, f_j\rangle = 0, \quad \langle f_j, f_k\rangle = \delta_{jk}, \quad (j, k = 2, \ldots, n). \tag{4.3}$$

Let us note that the basis vectors $f_2, \ldots, f_n$ are space-like, since they are orthogonal to the time-like vector u_d. The vector $x \in R^n$ decomposes as follows

$$x = \langle u_d, x\rangle \frac{u_d}{\langle u_d, u_d\rangle} + \sum_{j=2}^{n} \langle f_j, x\rangle f_j. \tag{4.4}$$

Hence, the basic vector equation given in Eq. (4.1) reads in term of coordinates in such a basis as follows

$$\langle f_j, \ddot{x}\rangle + \left(\langle u_d, \dot{x}\rangle - \dot{\gamma} + 2\eta\kappa^2 \mathrm{e}^{\gamma}\right)\langle f_j, \dot{x}\rangle = 0 \tag{4.5}$$

$$\frac{\langle u_d, \ddot{x}\rangle}{\langle u_d, u_d\rangle} + \left(\frac{\langle u_d, \dot{x}\rangle - \dot{\gamma}}{\langle u_d, u_d\rangle} + \zeta\kappa^2 \mathrm{e}^{\gamma}\right)\langle u_d, \dot{x}\rangle = \frac{h}{2}\left[\frac{\langle u_d, \dot{x}\rangle^2}{\langle u_d, u_d\rangle} + \sum_{j=2}^{n}\langle f_j, \dot{x}\rangle^2\right], \tag{4.6}$$

which provides us with a set of equations (for $j = 2, \ldots, n$). For the metric dependence of the viscosity coefficients, see Eqs. (2.25),(2.26) Eqs. (4.5,4.6) read

$$\langle f_j, \ddot{x}\rangle + \eta_0\langle f_j, \dot{x}\rangle = 0, \quad (j = 2, \ldots, n) \tag{4.7}$$

$$\langle u_d, \ddot{x}\rangle - \frac{h}{2}\langle u_d, \dot{x}\rangle^2 + \langle u_d, u_d\rangle \left(\zeta_0\langle u_d, \dot{x}\rangle - \frac{h}{2}\sum_{j=2}^{n}\langle f_j, \dot{x}\rangle^2\right) = 0 \tag{4.8}$$

in the harmonic time gauge

$$\gamma = \gamma_0 = \langle u_d, x\rangle = \sum_{i=1}^{n} N_i x^i. \tag{4.9}$$

Such a set of equations is integrable for any values of the constant parameters h, η_0 and ζ_0.

Let us first assume models with

$$h \neq 0. \tag{4.10}$$

The integration of Eq. (4.7) gives

$$\langle f_j, x\rangle = \begin{cases} tp^j + q^j & \text{if } \eta_0 = 0, \\ \mathrm{e}^{-\eta_0 t}p^j + q^j & \text{if } \eta_0 \neq 0, \end{cases} \tag{4.11}$$

where p^j and q^j are arbitrary constants. The Kasner-like form solution can be written in term of vectors $\alpha, \beta \in R^n$, defined as follows

$$\alpha = \sum_{j=2}^{n} p^j f_j \equiv \sum_{i=1}^{n} \alpha^i e_i, \quad \beta = \sum_{j=2}^{n} q^j f_j \equiv \sum_{i=1}^{n} \beta^i e_i, \tag{4.12}$$

where α^i and β^i are their coordinates in the canonical basis $e_1, \ldots, e_n$. By using the orthogonality conditions, we obtain

$$\langle \alpha, u_d \rangle = \sum_{i=1}^{n} \alpha^i N_i = 0, \quad \langle \beta, u_d \rangle = \sum_{i=1}^{n} \beta^i N_i = 0, \tag{4.13}$$

$$\langle \alpha, \alpha \rangle = \sum_{i=1}^{n} \left(\alpha^i\right)^2 N_i = \sum_{j=2}^{n} \left(p^j\right)^2 \geq 0 \tag{4.14}$$

$$\langle \beta, \beta \rangle = \sum_{i=1}^{n} \left(\beta^i\right)^2 N_i = \sum_{j=2}^{n} \left(q^j\right)^2 \geq 0, \tag{4.15}$$

where the constants α^i and β^i may be called Kasner-like parameters, because of the existence of these constraints. By using Eqs. (4.4,4.11,4.12), we obtain

$$x = \langle u_d, x \rangle \frac{u_d}{\langle u_d, u_d \rangle} + a(t)\alpha + \beta, \tag{4.16}$$

where the function

$$a(t) = \begin{cases} t & \text{if } \eta_0 = 0, \\ \mathrm{e}^{-\eta_0 t} & \text{if } \eta_0 \neq 0, \end{cases} \tag{4.17}$$

By substituting the functions $\langle f_j, x \rangle$ into Eq. (4.8) we obtain the following equation for the unknown function $\langle u_d, x \rangle$

$$\langle u_d, \ddot{x} \rangle - \frac{h}{2} \langle u_d, \dot{x} \rangle^2 + \langle u_d, u_d \rangle \zeta_0 \langle u_d, \dot{x} \rangle = -\frac{h}{2} A^2 \dot{a}^2(t), \tag{4.18}$$

where

$$A = \sqrt{\frac{D-1}{D-2} \langle \alpha, \alpha \rangle}. \tag{4.19}$$

The equation Eq. (4.18) has been integrated for the non-viscous model $\eta_0 = \zeta_0 = 0$ (see, e.g., [14]) and for the model with $\zeta_0 \neq 0$ and $\eta_0 = 0$ in [15, 17]. For the model with $\eta_0 \neq 0$, it can be reduced to the modified Bessel equation

$$\tau^2 \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + \tau \frac{\mathrm{d}z}{\mathrm{d}\tau} - (\tau^2 + \nu^2) z = 0 \tag{4.20}$$

by means of the transformation

$$\tau = \frac{1}{2} |h| A \mathrm{e}^{-\eta_0 t}, \tag{4.21}$$

$$\langle u_d, \dot{x} \rangle = \frac{2\eta_0}{h} \tau \frac{\mathrm{d}}{\mathrm{d}\tau} \ln |\tau^{-\nu} z(\tau)| \tag{4.22}$$

in the non-trivial case $\langle \alpha, \alpha \rangle \neq 0$, where the constant

$$\nu = \frac{D-1}{D-2} \frac{\zeta_0}{2\eta_0}. \tag{4.23}$$

The general solution of the modified Bessel equation is given by

$$z(\tau) = C_1 I_{|\nu|}(\tau) + C_2 K_{|\nu|}(\tau), \tag{4.24}$$

see [1], where $I_{|\nu|}(\tau)$, resp. $K_{|\nu|}(\tau)$, is the related modified Bessel function, resp. Mac-Donald function.

Finally, the results of Eqs. (4.7,4.8) integration for various constants ζ_0 and η_0 can be presented in Tab. 1

Table 1

	$\zeta_0 = 0$	$\zeta_0 \neq 0$
$\eta_0 = 0$	Solution II with $\langle \alpha, \alpha \rangle \neq 0$ and Solution I	Solution II
$\eta_0 \neq 0$	Solution III with $\langle \alpha, \alpha \rangle \neq 0$ and Solution I	Solution III with $\langle \alpha, \alpha \rangle \neq 0$ and Solution II with $\langle \alpha, \alpha \rangle = 0$

The solutions I,II,III in term of scale factors are the following

- Solution I :

$$e^{x^i} = e^{\beta^i} \left| C_1 + C_2 t \right|^{-2/[h(D-1)]}. \tag{4.25}$$

- Solution II :

$$e^{x^i} = e^{\alpha^i t + \beta^i} \left| C_1 e^{-(\tilde{A} - \tilde{\zeta}_0) h t/2} + C_2 e^{(\tilde{A} + \tilde{\zeta}_0) h t/2} \right|^{-2/[h(D-1)]}, \tag{4.26}$$

  where

$$\tilde{\zeta}_0 = \frac{D-1}{D-2} \frac{\zeta_0}{h}, \quad \tilde{A} = \sqrt{\frac{D-1}{D-2} \langle \alpha, \alpha \rangle + \left(\frac{D-1}{D-2} \frac{\zeta_0}{h} \right)^2} = \sqrt{A^2 + \tilde{\zeta}_0^2}. \tag{4.27}$$

- Solution III :

$$e^{x^i} = \exp\left(\alpha^i e^{-\eta_0 t} + \beta^i \right) \left(\tau^{-\nu} \left| C_1 I_{|\nu|}(\tau) + C_2 K_{|\nu|}(\tau) \right| \right)^{-2/[h(D-1)]}, \tag{4.28}$$

  where the variable $\tau > 0$ is given in Eq. (4.21), the constants A in Eq. (4.19) and ν in Eq. (4.23).

In formulas (4.25),(4.27),(4.28) $C_{i=1,2}$ are integration constants such that $C_1^2 + C_2^2 > 0$ and $i = 1, \dots, n$. The Kasner-like parameters α^i and β^i obey the relations given in Eq. (4.13).

The set of equations given in Eqs. (4.7,4.8) is easily integrable in the case

$$h = 0, \tag{4.29}$$

which, with the barotropic equation of state in mind, relates to Zeldovich or stiff matter. The results are given as follows :

- Solution IV : for $i = 1, \dots, n$

$$\mathrm{e}^{x^i} = \exp\left[\alpha^i a(t) + \beta^i\right] \times \begin{cases} \exp\left(C_1 + C_2 t\right) & \text{if } \zeta_0 = 0 \\ \exp\left(C_1 + C_2 \exp\left(\zeta_0 \frac{D-1}{D-2} t\right)\right) & \text{if } \zeta_0 \neq 0, \end{cases} \tag{4.30}$$

where $C_{i=1,2}$ are arbitrary constants, and the function $a(t)$ is given in Eq. (4.17).

5 Discussion

Let us remind the multidimensional generalization of the well-known *Kasner solution* [24], it reads (for the synchronous time t_s) as follows

$$\mathrm{d}s^2 = -\mathrm{d}t_s^2 + \sum_{i=1}^{n} A_i t_s^{2\varepsilon^i} \mathrm{d}s_i^2. \tag{5.1}$$

Such a metric describes the evolution of a vacuum model defined on the manifold $R \times M_1 \times \ldots \times M_n$, where the M_i are Ricci-flat spaces of dimension N_i with the metric $\mathrm{d}s_i^2$, A_i are arbitrary constants and ε_i are the Kasner parameters, which satisfy the relations

$$\sum_{i=1}^{n} N_i \varepsilon^i = 1, \quad \sum_{i=1}^{n} N_i (\varepsilon^i)^2 = 1. \tag{5.2}$$

The ε^i and the Kasner-like parameters α^i (used in the above formulas for the exact solutions) are related as

$$\varepsilon^i = \pm\frac{\alpha^i}{A} + \frac{1}{D-1}, \quad \langle \alpha, \alpha \rangle \neq 0. \tag{5.3}$$

By using Eq. (4.16) (i.e., a general decomposition of the vector $x \in R^n$) and the result of Sec.4, see Eq. (3.8), we obtain the variation rate of entropy

$$\dot{s} = \frac{B}{\kappa^2} \mathrm{e}^{-h\gamma_0} \left(\zeta_0 \dot{\gamma}_0^2 + \eta_0 \langle \alpha, \alpha \rangle \dot{a}^2(t)\right), \tag{5.4}$$

which shows that the entropy increases when $\zeta_0 dt > 0$ and $\eta_0 dt > 0$. Further we assume

$$\zeta_0 \geq 0, \quad \eta_0 \geq 0, \tag{5.5}$$

so harmonic time t (as well as synchronous time t_s) increases during the evolution.

One easily shows that the *weak energy condition* $T^{\mu}_{\nu} v^{\nu} v_{\mu} \geq 0$, for any D-dimensional non space-like vector v^{ν} (and thus as well as for the 4-dimensional case), applied to the stress-energy tensor given in Eq. (2.9), can be written as inequalities

$$\rho \geq 0, \quad \rho + p_i^* \geq 0, \quad (i = 1, \dots, n), \tag{5.6}$$

where ρ, resp. p_i^*, is the density, resp. the effective pressure, of the fluid.

The *dominant energy condition* $T^{\mu}_{\nu} v^{\nu} v_{\mu} \geq 0$ and $T^{\nu}_{\mu} T^{\mu}_{\lambda} v_{\nu} v^{\lambda} \leq 0$, for any non space-like vector v^{μ}, applied to the stress-energy tensor given in Eq. (2.9), reduces to inequalities defined in Eq. (5.6), with the following additional condition

$$\rho - p_i^* \geq 0, \quad (i = 1, \dots, n) \tag{5.7}$$

Notice that due to the weak energy condition the following restriction on the constant h (taken from the barotropic equation of state (2.11)) arises in the nonviscous case: $h \leq 2$. The dominant energy condition for the nonviscous stress-energy tensor implies: $0 \leq h \leq 2$.

It is important to note that Solution I and Solutions II,IV for zero Kasner-like parameters ($\alpha^i = 0, \quad i = 1, \dots, n$) are isotropic, since the spaces $M_1, \dots, M_n$ have identical scale factors.

Further we discuss the solutions obtained, which are of interest within multidimensional or 4-dimensional cosmology.

5.1 Non Viscous Models

For a better understanding of the viscosity effect on the dynamics, we first outline the properties of non viscous models.

5.1.1

The *isotropic non viscous model* is described by Solution I ($h \neq 0$) and Solution IV ($h = 0$) with $\alpha^i = 0$ (for $i = 1, \dots, n$), which represents the multidimensional generalization of the flat FRW model. It is the steady-state model

$$\mathrm{e}^{x^i} \sim \exp\left[-\frac{C_2}{D-1} t_s\right], \quad \rho = \mathrm{const} \tag{5.8}$$

for $h = 2$ and shows a power-law behavior

$$\mathrm{e}^{x^i} \sim t_s^{2/[(2-h)(D-1)]}, \quad \rho \sim t_s^{-2} \tag{5.9}$$

for $h \neq 2$, where t_s is the synchronous time (stationary solution with zero density is also possible); let us call it as Friedman-like behavior.

5.1.2

The *anisotropic non viscous model* for $h \in (0, 2]$ is described by Solution II with $\zeta_0 = 0$ and $\langle \alpha, \alpha \rangle \neq 0$. One easily shows that the integration constants $C_{i=1,2}$ have to satisfy the condition $C_1 C_2 < 0$, otherwise $\rho \leq 0, \quad \forall t$. By using this condition, we obtain from Eq. (4.26)

$$\mathrm{e}^{\gamma_0} \sim |\sinh[Ah(t-t_0)/2]|^{-2/h} \tag{5.10}$$

where $t_0 = 0$ can be chosen (with no loss of generality). Then, for a suitable choice of the integration constant of equation $\mathrm{d}t_s = \exp(\gamma_0)\mathrm{d}t$, one has the following correspondences $t \in (-\infty, 0) \Leftrightarrow t_s \in (0, +\infty)$ and $t \in (0, +\infty) \Leftrightarrow t_s \in (-\infty, 0)$. Hence, we solely investigate the solution $t_s \in (-\infty, 0)$, since the evolution of the non-viscous fluid is reversible.

From Eq. (5.10), we obtain in the main order $\exp[\gamma_0] \sim t^{-2/h}$ when $t \to +0$ ($t_s \to +\infty$), then one has $t_s \sim t^{1-2/h}$ for $h \in (0, 2)$ and $t_s \sim \ln t$ for $h = 2$. By using these relations and Eq. (4.26), we easily see that the multidimensional Universe shows an isotropical Friedman-like contraction, as defined by Eq. (5.9), in the (infinite) past. Such a conclusion is also valid for Zeldovich matter ($h = 0$).

Let us now investigate the behavior of the non-viscous anisotropic model at $t_s = 0$. For Solution II ($h \neq 0$), by using Eq. (5.10), we obtain in the main order $\exp[\gamma_0] \sim \exp[-At]$ when $t \to +\infty$ ($t_s \to +0$), then $t_s \sim \exp[-At]$. By substituting the latter relation into Eqs. (4.26), we obtain in the main order

$$\mathrm{e}^{x^i} \sim |t_s|^{-\alpha^i/A+1/(D-1)}, \quad \rho \sim |t_s|^{h-2} \quad (t_s \to +0). \tag{5.11}$$

According to Eq. (5.3), the model for $h \in (0, 2]$ has a Kasner-like behavior near the singularity (at $t_s = 0$). Such a behavior describes the contraction of some spaces $M_1, \ldots, M_n$ and the expansion for the other ones. According to Eq. (5.2), the number of either contracting or expanding spaces depends on n (the total number of spaces) and $N_{i=1,n}$ (their dimensions), but there is at least a contracting manifold and expanding one. Such dynamics is a mechanism of extra dimensions compactification (within the multidimensional cosmology).

One can easily show that the non-viscous anisotropic model for Zeldovich matter ($h = 0$) described by Solution IV has almost the same properties but with the second constraint given Eq. (5.2) for the Kasner parameters substituted by $\sum_{i=1}^{n} N_i(\varepsilon^i)^2 = \varepsilon$, where ε is a constant such that $1/(D-1) < \varepsilon < 1$.

5.2 Models with bulk viscosity

Let us now investigate the models with bulk viscosity.

5.2.1

The *viscous isotropic model* shows interesting features. The shear viscosity is not significant in this case and the model is described for $h \neq 0$ by Solution II with $\alpha^i = 0$ for $i = 1, \ldots, n$. If $C_1 C_2 < 0$ and $h > 0$ then the solution can be written as follows : for $i = 1, \ldots, n$

$$\mathrm{e}^{x^i} = R_i \left(1 - \exp\left[\frac{D-1}{D-2}\zeta_0(t-t_0)\right]\right)^{-2/[h(D-1)]}, \tag{5.12}$$

$$p_i^* = \left(1 - h\exp\left[-\frac{D-1}{D-2}\zeta_0(t-t_0)\right]\right)\rho, \tag{5.13}$$

$$\rho = \frac{D-1}{D-2}\frac{2\zeta_0^2}{\kappa^2 h^2}\prod_{i=1}^{n} R_i^{-2N_i} \exp\left[2\frac{D-1}{D-2}\zeta_0(t-t_0)\right]$$

$$\times\left(1-\exp\left[\frac{D-1}{D-2}\zeta_0(t-t_0)\right]\right)^{2(2-h)/h]}, \quad (5.14)$$

$$s = \frac{D-1}{D-2}\frac{2B\zeta_0^2}{\kappa^2 h^2}\prod_{i=1}^{n} R_i^{-hN_i}\exp\left[2\frac{D-1}{D-2}\zeta_0(t-t_0)\right]+s(-\infty), \quad (5.15)$$

where we use the set of independent constants $t_0, R_1, \dots, R_n$, defined such that

$$\frac{C_1}{C_2} = -\exp\left[\frac{D-1}{D-2}\zeta_0 t_0\right], \quad |C_1|^{-2/[h(D-1)]}\exp[\beta^i] = R_i. \quad (5.16)$$

Let us consider this solution on the interval $(-\infty, t_0)$ for $h \in (0,2)$. Then the synchronous time t_s changes during the evolution on the interval $(-\infty, +\infty)$. Such a solution is nonsingular and describes a monotonic isotropic expansion of the D-dimensional Universe with Friedman-like stage as $t_s \to +\infty$. The density ρ increases from zero in the infinite past to some maximum value and then decreases to zero at the Friedman-like stage. The maximum of the density is reached at $t \equiv t_m = (D-2)\ln[h/2]/[\zeta_0(D-1)] + t_0$, when $p_i^* = -\rho$, see Eq. (5.13). We have $p_i^* + \rho < 0$ for $t \in (-\infty, t_m)$ and $p_i^* + \rho > 0$ for $t \in (t_m, t_0)$, so the weak energy condition, see Eq. (5.6), is not satisfied on the time interval $(-\infty, t_m)$. The entropy per baryon monotonically increases during the evolution and tends to some constant value in the infinite future.

The nonsingular solution obtained by Murphy [35] within flat FRW model with bulk viscosity for another second equation of state: $\zeta = \mathrm{const}\rho$ exhibits the similar properties except the violating of the weak energy condition. The scale factor monotonically increases from zero in the infinite past and tends to the infinity at the Friedman stage of the evolution. The density monotonically decreases from some constant value in the infinite past and tends to zero according to Eq. (5.9) in the infinite future. The weak energy condition is valid for $t_s \in (-\infty, +\infty)$ (but the strong energy condition $\rho + 3p^* \geq 0$ is not satisfied on the interval $(-\infty, t_s^*)$, where t_s^* is some constant), so from this point of view the Murphy solution is more attractive.

5.2.2

Now, let us study the properties of the *anisotropic viscous model* by taking into account only the bulk viscosity. Such model is described by Solution II with non-zero Kasner-like parameters α^i and has been previously studied [15, 17]. If $C_1C_2 > 0$ in Solution II with $\alpha^i \neq 0$ then the density has negative values at some stage of the evolution, which means that the weak energy condition is not satisfied. Hence, hereafter such solutions are not considered. If $C_1C_2 < 0$, then Solution II can be written as follows : for $i = 1, \dots, n$

$$\mathrm{e}^{x^i} = R_i\left|\sinh[\tilde{A}h(t-t_0)/2]\right|^{-2/[h(D-1)]}\exp\left[\left(\alpha^i - \frac{\zeta_0}{h(D-2)}\right)t\right], \quad (5.17)$$

$$p_i^* = \left(1 - \frac{h\cosh^2\tilde{a}}{1+\sinh\tilde{a}\sinh[\tilde{A}h(t-t_0)+\tilde{a}]}\right)\rho, \quad (5.18)$$

$$\rho = \frac{\langle\alpha,\alpha\rangle}{2\kappa^2}\prod_{i=1}^{n} R_i^{-2N_i}\left|\sinh[\tilde{A}h(t-t_0)/2]\right|^{2(2-h)/h}\exp[2\tilde{\zeta}_0 t]$$

$$\times \left(1+\sinh\tilde{a}\sinh[\tilde{A}h(t-t_0)+\tilde{a}]\right), \tag{5.19}$$

$$\dot{s} = \frac{B\zeta_0 A^2}{\kappa^2}\prod_{i=1}^{n} R_i^{-hN_i}\exp\left[\frac{D-1}{D-2}\zeta_0\left(1-\frac{2}{h}\right)t\right]\sinh^2[\tilde{A}h(t-t_0)]$$

$$\times \sinh^2[\tilde{A}h(t-t_0)+\tilde{a}], \tag{5.20}$$

where we use independent constants $t_0, R_1, \ldots, R_n$ defined by

$$C_1 = -\frac{C}{2}\exp[\tilde{A}ht_0/2], \quad C_2 = \frac{C}{2}\exp[-\tilde{A}ht_0/2], \tag{5.21}$$

$$R_i = |C|^{-2/[h(D-1)]}\exp[\beta^i], \tag{5.22}$$

and $\tilde{a}$ is defined by

$$\sinh\tilde{a} = \frac{\tilde{\zeta}_0}{A}, \quad \cosh\tilde{a} = \frac{\tilde{A}}{A}. \tag{5.23}$$

Let us note that for $t\in(-\infty,t_0)$ the density, given by Eq. (5.19), has negative values. In the following, this solution is only used on the interval $(t_0,+\infty)$.

If $h\in(0,2)$ then the harmonic time interval $t\ :\ (t_0,+\infty)$ corresponds to the following synchronous time interval $t_s\ :\ (-\infty,t_s^0)$, where we choose the integration constant $t_s^0=0$. We can easily prove that such a model has the stage of isotropic contraction by the Friedman-like law defined in Eq. (5.9) in the infinite past. Near the final point of the evolution $t_s=0$ we obtain in the main order

$$\mathrm{e}^{x^i} \sim t_s^{-\alpha^i/(\tilde{A}+\tilde{\zeta}_0)+1/(D-1)}, \ i=1,\ldots,n. \tag{5.24}$$

The final point of the evolution $t_s=0$ is singular ($\rho\to+\infty$ as $t_s\to-0$), and the characteristic of the singularity depends on the parameter $\tilde{\zeta}_0/A$ (which determines the ratio between the viscosity parameter $\tilde{\zeta}_0$ and the anisotropy parameter A). If the ratio $\tilde{\zeta}_0/A\gg1$ then we obtain from Eq. (5.24) $\exp[x^i]\sim t_s^{1/(D-1)}$ near $t_s=0$. In such a case, the model describes the contraction of all spaces $M_1,\ldots,M_n$ in the vicinity of the singularity. If $\tilde{\zeta}_0/A\ll1$ then we obtain ¿from Eq. (5.24) $\exp[x^i]\sim t_s^{\varepsilon^i}$ (i.e. the singularity is of the Kasner type). According to Eq. (5.20), in both cases the model describes the unbounded production of the entropy at the final stage of the evolution.

Therefore, anisotropic Solution II for $h\in(0,2)$ and $C_1C_2<0$ describes the model with Friedman-like isotropic contraction in the infinite past and the anisotropic Kasner-like behavior near the final singularity if the parameter $\tilde{\zeta}_0/A$ is small enough. Under such a condition, the behavior of scale factors remains (qualitatively) the same as for the anisotropic non-viscous model for $t_s<0$. Let us note that this model satisfies the dominant energy condition during the evolution. According to Eq. (5.18), the ratio p_i^*/ρ increases monotonically from the value $(1-h)$ at the Friedman-like stage in the infinite past, and tends to 1 in the vicinity of the final singularity. One may also consider Solution II for $h>0$ and $C_1=0$, $C_2\neq0$. This partial solution has the similar dynamical behavior at the final stage of the evolution and satisfies the dominant energy condition as $p_i^*=\rho$, $\forall t_s$.

If the dominant energy condition is ignored then we may also consider anisotropic Solution II for $h<0$ and $C_1C_2<0$. In such a case the harmonic time interval $t\ :\ (t_0,+\infty)$ corresponds to the synchronous time interval $t_s\ :\ (0,+\infty)$. We easily show that such a model has the Friedman-like singularity, defined in Eq. (5.9) at $t_s = 0$. The anisotropic behavior is possible far from the singularity, if the parameter $|\tilde{\zeta}_0|/A$ is small enough. This solution satisfies the weak energy condition as the ratio p_i^*/ρ monotonically decreases during the evolution from the value $(1-h)$ at the Friedman-like stage, and tends to 1 in the infinite future. Let us note that the asymptotical behavior of this solution far from the singularity ($t_s \to +\infty$) is given by the Solution II for $h < 0$ and $C_2 = 0$, $C_1 \neq 0$. To investigate the possible anisotropic behavior far from the singularity let us consider this partial solution. It may be written in the synchronous time as following : for $i = 1, \ldots, n$

$$\begin{aligned} \mathrm{e}^{x^i} &= R_i t_s^{\alpha^i/(\tilde{A}+|\tilde{\zeta}_0|)+1/(D-1)}, \quad t_s > 0, & (5.25)\\ p_i^* = \rho &= \frac{D-2}{D-1}\frac{|\tilde{\zeta}_0|}{\kappa^2(\tilde{A}+|\tilde{\zeta}_0|)} t_s^{-2}, & (5.26)\\ s &= \frac{(D-2)B|\zeta_0|(\tilde{A}+|\tilde{\zeta}_0|)}{(D-1)\kappa^2} t_s^{-h} + s(0), & (5.27) \end{aligned}$$

where the constants $R_1, \ldots, R_n$ are such that

$$\prod_{i=1}^{n} R_i^{N_i} = \tilde{A} + |\tilde{\zeta}_0|. \quad (5.28)$$

It is evident from Eq. (5.25) that $\alpha^i/(\tilde{A} + |\tilde{\zeta}_0|) + 1/(D-1) \to \varepsilon^i$ (Kasner parameter) as $|\tilde{\zeta}_0|/A \to 0$. Therefore, if the parameter $|\tilde{\zeta}_0|/A$ is small enough, then the model describes the contraction of a part of the spaces $M_1, \ldots, M_n$ and the expansion of another part. According to Eq. (5.27), the solution describes the unbounded entropy production as $h < 0$.

We note, that the anisotropic viscous model described by Solution IV ($h = 0$) does not satisfy the weak energy condition as the density has negative values at some stage of the evolution.

5.3 Models with bulk and shear viscosity

Now let us investigate the model with both bulk and shear viscosity. Such a model for $h \neq 0$ is described by Solution III with non-zero Kasner like parameters α^i and $\nu > 0$. By using the following properties of the modified Bessel functions [1]

$$I_{\nu+1}(\tau) < I_\nu(\tau), \quad K_{\nu+1}(\tau) > K_\nu(\tau) \quad \forall \tau \in (0, +\infty), \nu \geq 0, \quad (5.29)$$

it can be proved that the density given has negative values at some interval of time if $C_2 = 0$. If $C_2 \neq 0$ then the Solution III can be written as follows: for $i = 1, \ldots, n$

$$\begin{aligned} \mathrm{e}^{x^i} &= R_i \left(\tau^{-\nu}\left|CI_\nu(\tau) + K_\nu(\tau)\right|\right)^{-2/[h(D-1)]} \exp\left[\alpha^i \mathrm{e}^{-\eta_0 t}\right], & (5.30)\\ \dot{x}^i &= \frac{2\eta_0\tau}{h}\left[\left(\frac{CI_{\nu+1}(\tau) - K_{\nu+1}(\tau)}{CI_\nu(\tau) + K_\nu(\tau)} + 1\right)\frac{1}{D-1} - \varepsilon^i\right], & (5.31)\\ p_i^* &= \left(1 - h + \frac{h}{\tau}\frac{(D-1)\varepsilon^i - 1 - \nu\frac{CI_{\nu+1}(\tau)-K_{\nu+1}(\tau)}{CI_\nu(\tau)+K_\nu(\tau)}}{\left(\frac{CI_{\nu+1}(\tau)-K_{\nu+1}(\tau)}{CI_\nu(\tau)+K_\nu(\tau)}\right)^2 - 1}\right)\rho, & (5.32) \end{aligned}$$

$$\begin{aligned}
\rho &= \frac{2\eta_0^2}{h^2\kappa^2}\frac{D-2}{D-1}\prod_{i=1}^{n} R_i^{-2N_i}\tau^{2(1-2\frac{\nu}{h})}\left|CI_\nu(\tau)+K_\nu(\tau)\right|^{\frac{4}{h}} \\
&\times\left[\left(\frac{CI_{\nu+1}(\tau)-K_{\nu+1}(\tau)}{CI_\nu(\tau)+K_\nu(\tau)}\right)^2-1\right], \qquad (5.33) \\
\dot{s} &= \frac{4\eta_0^3 B}{h^2\kappa^2}\frac{D-1}{D-2}\prod_{i=1}^{n} R_i^{-hN_i}\tau^{2(1-\nu)} \\
&\times\left(\nu[CI_{\nu+1}(\tau)-K_{\nu+1}(\tau)]^2+[CI_\nu(\tau)+K_\nu(\tau)]^2\right), \qquad (5.34)
\end{aligned}$$

where the independent constants $R_1, \ldots, R_n, C = C_1/C_2$ are used. The variable τ was determined in Eq. (4.21). Also we introduced Kasner parameters ε^i by

$$\varepsilon^i = \mathrm{sgn}[h]\frac{\alpha^i}{A} + \frac{1}{D-1}. \qquad (5.35)$$

By using the properties given in Eq. (4.29) and the asymptotical behavior of the modified Bessel functions [1], one may prove that the density given in Eq. (5.33) has no negative values during the evolution only for :

1. partial solution with $C = 0$ on the interval $(-\infty, +\infty)$ of the harmonic time;
2. solution with $C < 0$ on the interval $(t_0(\tau_0), +\infty)$, where τ_0 is the root of the equation

$$CI_\nu(\tau_0) + K_\nu(\tau_0) = 0. \qquad (5.36)$$

We note that the asymptotical behavior of the solution with $C < 0$ as $t \to +\infty$ is given by the partial solution with $C = 0$.

Let us now investigate the models having a singularity at the beginning of the evolution. Such solutions arise when $h < 0$. In this case for the solutions with $C < 0$ one obtain the following correspondences $(-\tau) \in (-\tau_0, 0) \Leftrightarrow t \in (t_0(\tau_o), +\infty) \Leftrightarrow t_s \in (0, +\infty)$. The singularity at $t_s = 0$ is of Friedman type. The shear viscosity leads to the isotropization at the final stage of evolution. We obtain from Eq. (5.30-5.33) $\exp[x^i] \sim t_s^{1/(D-1)}$ and $p_i^* = \rho \sim t_s^{-2}$ as $t_s \to +\infty$, i.e. the model describes the isotropic Friedman-like expansion corresponding to Zeldovich matter ($h = 0$). The anisotropic behavior is possible on some interval of time after the Friedman-like behavior. One can prove that if the constant $|C| \neq 0$ is small enough then on some interval the sign of the Hubble parameter $\dot{x}^i$ coincides with the sign of the Kasner parameter ε^i (see Eq. (5.31) for $h < 0$). However, such regime is possible only on limited interval of time, because the final stage of the evolution exhibits the isotropic expansion due to the shear viscosity.

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